

UNBOUNDED ASYMMETRY OF STRETCH FACTORS

SPENCER DOWDALL, ILYA KAPOVICH, AND CHRISTOPHER J. LEININGER

ABSTRACT. A result of Handel–Mosher guarantees that the ratio of logarithms of stretch factors of any fully irreducible automorphism of the free group F_N and its inverse is bounded by a constant C_N . In this short note we show that this constant C_N cannot be chosen independent of N .

Let F_N be the free group of rank $N \geq 2$. An outer automorphism $\varphi \in \text{Out}(F_N)$ is said to be *fully irreducible* if no power of φ preserves the conjugacy class of any proper free factor of F_N . In this case φ has a well defined *stretch factor* $\lambda(\varphi)$, which, for any non- φ -periodic conjugacy class α in F_N and a free basis X of F_N , is given by

$$\lambda(\varphi) = \lim_{n \rightarrow \infty} \sqrt[n]{\|\varphi^n(\alpha)\|_X},$$

where $\|\cdot\|_X$ denotes cyclically reduced word length with respect to X . As was observed in [BH] (see also [HM2]), there exist fully irreducible elements $\varphi \in \text{Out}(F_N)$ with the property that φ and φ^{-1} have *different* stretch factors:

$$\lambda(\varphi) \neq \lambda(\varphi^{-1}).$$

However, the following result from [HM1] describes the extent to which they can differ. To state their result precisely, let $N \geq 2$ and set

$$C_N = \sup_{\varphi} \frac{\log(\lambda(\varphi))}{\log(\lambda(\varphi^{-1}))},$$

where φ ranges over all fully irreducible elements of $\text{Out}(F_N)$.

Theorem 1 (Handel–Mosher). *For $N \geq 2$, $C_N < \infty$.*

An alternate proof of this result was more recently given by Algom-Kfir and Bestvina [AKB]. While the proofs of this theorem appeal to the fact that N is fixed, it is not clear that this dependence is necessary. In this short note, we prove that in fact it is.

Theorem 2. *With $\{C_N\}_{N \geq 2}$ defined as above, $\limsup_{N \rightarrow \infty} C_N = \infty$.*

Proof. The proof will appeal to a construction and analysis carried out in [DKL1] and [DKL2]. To that end, let $F_3 = \langle a, b, c \rangle$ and consider the element $\varphi \in \text{Aut}(F_3)$ defined by

$$\varphi(a) = b, \quad \varphi(b) = b^{-1}a^{-1}bac, \quad \varphi(c) = a.$$

It was shown in [DKL1, Example 5.5] that φ is fully irreducible. Next, let

$$G = F_3 \rtimes_{\varphi} \mathbb{Z} = \langle a, b, c, r \mid r^{-1}xr = \varphi(x) \text{ for all } g \in F_3 \rangle$$

be the free-by-cyclic group determined by φ , and let $u_0: G \rightarrow \mathbb{Z}$ in $\text{Hom}(G; \mathbb{R}) = H^1(G; \mathbb{R})$ be the associated homomorphism obtained by sending r to 1 and all other generators to 0.

In [DKL1], we construct a cone $\mathcal{A} \subset H^1(G; \mathbb{R})$ containing u_0 with the property that every other primitive integral element $u \in \mathcal{A}$ has kernel $\ker(u)$ a finitely generated free group. The action of $u(G) = \mathbb{Z}$ on $\ker(u)$ is generated by a *monodromy automorphism* $\varphi_u \in \text{Aut}(\ker(u))$ determining an expression of G as a semidirect

2010 *Mathematics Subject Classification.* Primary 20F65, Secondary 57M, 37B, 37E.

The first author was partially supported by the NSF postdoctoral fellowship, NSF MSPRF no. 1204814. The second author was partially supported by the NSF grant DMS-0904200 and by the Simons Foundation Collaboration grant no. 279836. The third author was partially supported by the NSF grant DMS-1207183. The third author acknowledges support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 “GEAR Network”.

product $G \cong \ker(u) \rtimes_{\varphi_u} \mathbb{Z}$ with associated homomorphism u . One of the main results of [DKL1] is that all such φ_u are fully irreducible.

In [DKL2], we construct a strictly larger open, convex cone $\mathcal{A} \subsetneq \mathcal{S} \subset H^1(G; \mathbb{R})$ and a function

$$\mathfrak{H}: \mathcal{S} \rightarrow \mathbb{R}$$

that is convex, real analytic, and homogeneous of degree -1 (i.e., $\mathfrak{H}(tu) = \frac{1}{t}\mathfrak{H}(u)$) such that

$$\log(\lambda(\varphi_u)) = \mathfrak{H}(u)$$

for any primitive integral class $u \in \mathcal{A}$. In fact this holds for all primitive integral $u \in \mathcal{S}$ with the appropriate interpretation of $\lambda(\varphi_u)$. We also show that \mathcal{S} is the cone on the component of the BNS-invariant $\Sigma(G)$ [BNS] containing u_0 [DKL2, Theorem I] and that \mathcal{A} lies over the symmetrized BNS-invariant (that is, both \mathcal{A} and $-\mathcal{A}$ project into $\Sigma(G)$) [DKL2, Corollary 13.7]. In fact, a key result of Bieri–Neumann–Strebel is that an integral class $u \in \text{Hom}(G; \mathbb{Z})$ has $\ker(u)$ finitely generated if and only if both u and $-u$ lie in the $\Sigma(G)$ [BNS].

The homomorphism $-u_0$ has $\ker(-u_0) = \ker(u_0) = F_N$ and associated monodromy φ^{-1} , thus expressing G as $F_N \rtimes_{\varphi^{-1}} \mathbb{Z}$. Since φ^{-1} is also fully irreducible, the main result of [DKL2] provides another open, convex cone $\mathcal{S}_- \subset H^1(G; \mathbb{R})$ containing $-u_0$ and a corresponding convex, real analytic, homogeneous of degree -1 function $\mathfrak{H}_-: \mathcal{S}_- \rightarrow \mathbb{R}$. Since $-\mathcal{A}$ projects into $\Sigma(G)$ and \mathcal{S}_- is the cone on the component of $\Sigma(G)$ containing $-u_0$, we see that $-\mathcal{A} \subset \mathcal{S}_-$. Thus \mathfrak{H}_- calculates the inverse stretch factors

$$\mathfrak{H}_-(-u) = \log(\lambda(\varphi_u^{-1}))$$

for all primitive integral $u \in \mathcal{A}$.

Example 8.3 of [DKL2] exhibits a primitive integral class $u_1 \in \mathcal{S}$ which lies on the boundary of \mathcal{A} (see [DKL2, Figure 8]) for which $\ker(u_1)$ is *not* finitely generated. It follows that $-u_1$ is *not* in the BNS-invariant. The key observation is that $-u_1$ then necessarily lies on the boundary of \mathcal{S}_- (since $-u_1 \in \overline{-\mathcal{A}} \subset \overline{\mathcal{S}_-}$ but $-u_1 \notin \mathcal{S}_-$). Let $\{u_n\}_{n=2}^\infty \subset \mathcal{A}$ be primitive integral classes protectively converging to u_1 . That is, there exists $\{t_n\}_{n=2}^\infty \subset \mathbb{R}$ so that $\lim_{n \rightarrow \infty} t_n u_n = u_1$. Since this convergence occurs inside \mathcal{S} , it follows that

$$\lim_{n \rightarrow \infty} \mathfrak{H}(t_n u_n) = \mathfrak{H}(u_1) < \infty.$$

On the other hand, since $\lim_{n \rightarrow \infty} -t_n u_n = -u_1 \in \partial \mathcal{S}_-$, it follows from [DKL2, Theorem F] that

$$\lim_{n \rightarrow \infty} \mathfrak{H}_-(-t_n u_n) = \infty.$$

Therefore, appealing to the homogeneity of \mathfrak{H} and \mathfrak{H}_- , we have

$$\lim_{n \rightarrow \infty} \frac{\log(\lambda(\varphi_{u_n}^{-1}))}{\log(\lambda(\varphi_{u_n}))} = \lim_{n \rightarrow \infty} \frac{\mathfrak{H}_-(-u_n)}{\mathfrak{H}(u_n)} = \lim_{n \rightarrow \infty} \frac{\mathfrak{H}_-(-t_n u_n)}{\mathfrak{H}(t_n u_n)} = \infty. \quad \square$$

REFERENCES

- [AKB] Yael Algom-Kfir and Mladen Bestvina. Asymmetry of outer space. *Geom. Dedicata*, 156:81–92, 2012.
- [BH] Mladen Bestvina and Michael Handel. Train tracks and automorphisms of free groups. *Ann. of Math. (2)*, 135(1):1–51, 1992.
- [BNS] Robert Bieri, Walter D. Neumann, and Ralph Strebel. A geometric invariant of discrete groups. *Invent. Math.*, 90(3):451–477, 1987.
- [DKL1] Spencer Dowdall, Ilya Kapovich, and Christopher J. Leininger. Dynamics on free-by-cyclic-groups. 2013. preprint arXiv:1301.7739.
- [DKL2] Spencer Dowdall, Ilya Kapovich, and Christopher J. Leininger. McMullen polynomials and Lipschitz flows for free-by-cyclic groups. 2013. preprint arXiv:1310.7481.
- [HM1] Michael Handel and Lee Mosher. The expansion factors of an outer automorphism and its inverse. *Trans. Amer. Math. Soc.*, 359(7):3185–3208 (electronic), 2007.
- [HM2] Michael Handel and Lee Mosher. Parageometric outer automorphisms of free groups. *Trans. Amer. Math. Soc.*, 359(7):3153–3183 (electronic), 2007.

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801
<http://www.math.uiuc.edu/~dowdall/>, <http://www.math.uiuc.edu/~kapovich/>, <http://www.math.uiuc.edu/~clein/>
E-mail address: dowdall@illinois.edu, kapovich@math.uiuc.edu, clein@math.uiuc.edu